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A New Method to Study Hawking Tunneling Radiation of the Charged Particles from Ressiner-Nordström Black Hole

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The tunneling radiation of Ressiner-Nordström black hole is studied by developing Hamilton-Jacobi method. The result shows the actual radiation spectrum deviates from the pure thermal one and the tunneling probability are related to the change of Bekenstein-Hawking entropy, which is accordant with Parikh and Wilczek's and gives a new method to correct Hawking pure thermal radiation of Ressiner-Nordström black hole.

KEY WORDS: Ressiner-Nordström black hole; self-gravitation interaction; Hamilton-Jacobi method; tunneling probability; Bekenstein-Hawking entropy.

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At the beginning of 1970s, Stephen Hawking made a striking discovery that a collapsing body radiates thermally and the derived radiation spectrum is pure thermal (Hawking, 1974, 1975). When proving the existence of black hole radiation, he described it as a quantum tunneling process triggered by vacuum fluctuations near the horizon. According to this scenario, the tunneling picture can be described as follows: a virtual particle pair creates spontaneously at a point just inside of the black hole horizon. The positive energy particle then tunnels out to the opposite of the horizon and materializes a real particle; while the negative energy particle remains behind and is absorbed by the black hole.

Basing on the picture, a new semi-classical tunneling method was initiated by Kraus and Wilczek and developed by Parikh and Wilczek (Kraus and Wilczek 1995; Parikh, 2000, 2002, 2004). The essence of it is a dynamical treatment of Hawking radiation of the black holes. To the methodology level, the energy conservation and the self-gravitation interaction are taken into account. Besides, the key point of the method is to find the motion equation of the emitted particle in Painlevé coordinate system and to calculate the action by Hamilton equation. In

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fact, the coordinate system regular at the horizon is quite appropriate to describe the Hawking radiation of a slowly evaporating black hole. They applied it to the Hawking radiation of the static Schwarzschild black hole and Ressiner-Nordström black hole, and the result shows the true radiation spectrum deviates from the pure thermal one and the tunneling probability are related to the change of Bekenstein-Hawking entropy. According to this method, quite a few fruits have been achieved and all of their results supported Parikh's opinion and even offered a correct amendment to Hawking pure thermal radiation (Hemming and Keski-Vakkuri, 2001; Vagenas, 2002; Medved, 2002, 2005; Zhang and Zhao, 2005a,b,c; Yang, 2005; Li *et al.*, 2005; Jiang *et al.*, 2006; Chen *et al.*, 2006). Zhang and Zhao extended the method to the Hawking radiation of the charged particle in 2005 and 2006. (Zhang and Zhao, 2005a,b,c, 2006). In summary, we have achieved success in the Hawking radiation of black holes by the method.

In the same year, Angheben *et al.* put forward another method and applied it to Hawking radiation as tunneling (Angheben *et al.*, 2005; Kerner and Mann, 2006). It is to calculate the classical action by Hamilton-Jacobi equation and referred to Hamilton-Jacobi method. The method can avoid performing Painlevé coordinate transformation and find the motion equation of the radiation particle. However, since the self-gravitation interaction of the emitted particle wasn't taken into account, the derived radiation spectrum is also pure thermal. In addition, for the rotating black holes, the dragging coordinate transformation wasn't performed in the process.

Thus our goal in this paper is to attempt to develop Hamilton-Jacobi method and to apply it to the study of Hawking radiation of the static Ressiner-Nordström black hole. The result shows, when taking the self-gravitation interaction as well as the conservation of energy and charge into account, the tunneling probability is related to the change of Bekenstein-Hawking entropy and the radiation spectrum deviates from pure thermal one, which is accordant with Parikh and Wilczek's and gives a correction to the Hawking radiation of Ressiner-Nordström black hole. Now, the Hawking radiation of the black hole by Hamilton-Jacobi method is discussed as follows.

The line element of Ressiner-Nordström black hole is given by

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (1)$$

with the electromagnetic potential $A_\mu = (A_t, 0, 0, 0)$ and the entropy $S = \pi r_h^2$, where

$$A_t = -\frac{Q}{r}, \quad \Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad (2)$$

and $r_h = M + \sqrt{M - Q^2}$ is the location of the event horizon. In Parikh and Wilczek's opinion, one should introduce Painlevé coordinate transformation to let the line element be well at the event horizon. And the motion equation of the particle should be also calculated in order to calculate the action. In fact, these can be avoided in Hamilton-Jacobi method. Near the event horizon of the black hole, we can get the line element as

$$ds^2 = -\Delta_{,r}(r_h)(r - r_h)dt^2 + [\Delta_{,r}(r_h)(r - r_h)]^{-1}dr^2 + r_h^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

where $\Delta_{,r}(r_h) = \frac{\partial\Delta}{\partial r}\big|_{r=r_h}$. Firstly we explore the classical action I of the charged particle, which satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}(\partial_\mu I - qA_\mu)(\partial_\nu I - qA_\nu) + u^2 = 0, \tag{4}$$

where u and q are the mass and charge of the emitted particle respectively, and $g^{\mu\nu}$ is the inverse metric derived from line element (3), substituting it into Eq. (4), we can get

$$\begin{aligned} &-\frac{1}{\Delta_{,r}(r_h)(r - r_h)}(\partial_t I - qA_t)^2 + \Delta_{,r}(r_h)(r - r_h)(\partial_r I)^2 \\ &+\frac{1}{r_h^2}\left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi I)^2\right] + u^2 = 0. \end{aligned} \tag{5}$$

It is difficult to calculate the action from Eq. (5) directly, considering the symmetry of the black hole, thus we carry on the following separation variable

$$I = -\omega t + R(r) + Y(\theta, \varphi), \tag{6}$$

where ω is the energy of the emitted particle, $R(r)$ is the generalized momentum in radial. From Eqs. (5) and (6), we can get

$$R(r) = \frac{1}{\Delta_{,r}(r_h)}\int \frac{dr}{r - r_h}\sqrt{(\omega + qA_t)^2 - \frac{\Delta_{,r}(r_h)(r - r_h)}{r_h^2}\left[\left(\partial_\theta Y + \frac{\partial_\varphi Y}{\sin^2\theta}\right) + u^2\right]}. \tag{7}$$

For getting the correct result, it is important to introduce the proper spatial distance (Angheben *et al.*, 2005; Kerner and Mann, 2006), which is defined by

$$d\sigma^2 = [\Delta_{,r}(r - r_h)]^{-1}dr^2 + r_h^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{8}$$

Limiting to the s-wave contribution that is contained the bulk of the particle emission, we can get

$$\sigma = \frac{2}{\sqrt{\Delta_{,r}(r_h)}}\sqrt{r - r_h}. \tag{9}$$

So Eq. (7) is rewritten as

$$R(\sigma) = \frac{2}{\Delta_{,r}(r_h)}\int \frac{d\sigma}{\sigma}\sqrt{(\omega + qA_t)^2 - \frac{\Delta_{,r}(r_h)(r - r_h)}{r_h^2}\left[\left(\partial_\theta Y + \frac{\partial_\varphi Y}{\sin^2\theta}\right) + u^2\right]}, \tag{10}$$

where the solution is singular at $\sigma = 0$, which corresponds to the event horizon. Thus, deforming the integration contour from the real σ-axis to the lower complex σ-plane that avoids the pole $\sigma = 0$ counterclockwise, and using the Feynman prescription at the event horizon, we can obtain the imaginary part of the action I as

$$\mathrm{Im} I = \frac{2\pi}{\Delta_{,r}(r_h)}\left(\omega - q\frac{Q}{r_h}\right). \tag{11}$$

Using WKB approximation (Kraus and Parentani, 2000), we can get the tunneling probability of the emitted particle and find the radiation spectrum being pure thermal. However, the recent research shows the radiation spectrum deviates from the pure thermal one and the tunneling probability are related to the change of Bekenstein-Hawking entropy. The reason of the pure thermal spectrum is that the self-gravitation interaction of the emitted particle was not considered in the process. Now taking the self-gravitation interaction as well as the conservation of energy and charge into account, we move on discussing the Hawking radiation of Ressiner-Nordström black hole. Fix the mass and charge of the total space-time and allow those of the black hole to be varied, when a particle with energy ω and charge q tunnels out, the parameters of the mass and charge in Eq. (11) should be changed and the correct imaginary part of the action is

$$\begin{aligned}\mathrm{Im} I &= \int_{(0,0)}^{(\omega,q)} \frac{2\pi}{\Delta'_{,r}(r'_h)}\left(d\omega' - \frac{(Q-q')}{r'_h}dq'\right) \\ &= -\int_{(M,Q)}^{(M-\omega,Q-q)} \frac{2\pi}{\Delta'_{,r}(r'_h)}\left[d(M-\omega') - \frac{(Q-q')}{r'_h}d(Q-q')\right],\end{aligned} \tag{12}$$

where

$$\Delta'_{,r}(r'_h) = \frac{2(M-\omega')}{r'^2_h} - \frac{2(Q-q')^2}{r'^3_h}, \quad r'_h = M - \omega' + \sqrt{(M-\omega')^2 - (Q-q')^2}. \tag{13}$$

Substituting Eq. (13) into (12), we can get

$$\begin{aligned}\mathrm{Im} I &= -\int_{(M,Q)}^{(M-\omega,Q-q)} \frac{\pi r'^3_h}{(M-\omega')r'_h - (Q-q')^2}\left[d(M-\omega') - \frac{Q-q'}{r'_h}d(Q-q')\right] \\ &= -2\pi\left[(M-\omega)^2 - \frac{(Q-q)^2}{2} + (M-\omega)\sqrt{(M-\omega)^2-(Q-q)^2} - M^2\right.\end{aligned}$$

$$+\frac{Q^2}{2} - M\sqrt{M^2 - Q^2}\Big]$$

$$= -\frac{1}{2}\left[S_{BH}(M-\omega, Q-q) - S_{BH}(M, Q)\right]. \tag{14}$$

Using WKB approximation (Kraus and Parentani, 2000), the tunneling probability of the charged particle can be obtained as

$$\Gamma \sim e^{-2\mathrm{Im}I} = e^{\Delta S_{BH}}, \tag{15}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, Q-q) - S_{BH}(M, Q)$ is the change of Bekenstein -Hawking entropy. The result shows the tunneling probability is related to the change of Bekenstein-Hawking entropy and the radiation spectrum deviates from pure thermal one, which is accordant with Parikh and Wilczek's.

Meanwhile, we can also get the same result from the thermodynamics properties of the black hole. The first law of the black hole thermodynamics tells us (Wu, 2005)

$$d(M-\omega') = \frac{1}{2\pi}\kappa' dS' + \phi'_h d\left(Q - q'\right), \tag{16}$$

where $\phi'_h = \frac{(Q-q')}{r'_h}$ and $\kappa' = \frac{\Delta'_{,r}(r'_h)}{2}$ are the electromagnetic potential and surface gravity after the particle emission. From Eq. (12) and (16), we can get the imaginary part of the action as

$$\mathrm{Im}I = -\pi \int\limits_{(M,Q)}^{(M-\omega,Q-q)} \frac{d(M-\omega') - \phi'_h d(Q-q')}{\kappa'}$$

$$= -\frac{1}{2} \int\limits_{S_{BH}(M,Q)}^{S_{BH}(M-\omega,Q-q)} dS' = -\frac{1}{2}\Delta S_{BH}. \tag{17}$$

Which also proves that the tunneling probability which is consistent with Eq. (15) and implies the correctness of our amendments to the Hawking radiation of Ressiner-Nordström black hole by developing Hamilton-Jacobi method.

In this paper, we have discussed the Hawking radiation of the charged particle from Ressiner-Nordström black hole by developing Hamilton-Jacobi method. Taking the self-gravitation interaction as well as the energy conservation and charge conservation, the radiation spectrum deviates from the pure thermal one and the tunneling probability are related to the change of Bekenstein-Hawking entropy. The result is full consistent with Parikh and Wilczek's, both of them give a correction to the Hawking pure thermal radiation of Ressiner-Nordström black hole.

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